A NONASPHERICAL CELL-LIKE 2-DIMENSIONAL SIMPLY CONNECTED CONTINUUM AND RELATED CONSTRUCTIONS

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Abstract. We prove the existence of a 2-dimensional nonaspherical simply connected cell-like Peano continuum (the space itself was constructed in one of our earlier papers). We also indicate some relations between this space and the well-known Griffiths' space from the 1950's.

1. Introduction

It is well-known (see [10, 12]) that every n-dimensional compactum is weakly homotopy equivalent to an $(n+1)$-dimensional cell-like compactum (i.e. a compactum with the trivial shape). Therefore there exist nonaspherical cell-like simply connected compacta in all dimensions ≥ 3.

It was heretofore unknown whether such compacta also exist in dimension 2. In this paper we give the affirmative answer to this question. We show that the space $SC(S^1)$ which we constructed in our earlier paper [9], is in fact, a *nonaspherical* cell-like 2-dimensional simply connected *Peano* continuum (i.e. locally connected continuum).

We also modify our original construction of the space $SC(S^1)$ and show that the modified construction gives a space which has the homotopy type of the classical well-known space [11] from the 1950's, which is a non-simply connected one-point union of two contractible spaces.

Our main result concerns $SC(X)$ for a non-simply connected path-connected space X. To analyze the sigular homology $H_2(SC(X))$, we use infinitary words and a result from [5]. Although infinitary words have already been introduced in [1], they may not be a familiar notion. In the special case $X = S^1$, we can prove the result only by using finitary words - we present it at the end of Section 3.

2. Preliminaries

We recall the construction of the space $SC(S^1)$ from [9]. Consider the so-called *Topologist sine curve* T and embed T into the square $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ as in Figure 1, i.e. T is embeded as the union of $A_1B_1A_2B_2\cdots$ and AB. Let S^1 be the circle and s_0 any of its points which we consider as the base point. Consider the topological sum of $\mathbb{I}^2$ and $T \times S^1$. The space $SC(S^1)$ is now defined as the quotient space of this sum, obtained by identification of the points (t, s_0) with $t \in T \subset \mathbb{I}^2$, and by identification of each set $\{t\} \times S^1$ with t, when $t \in \{0\} \times \mathbb{I}$. For an arbitrary

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compactum X, one defines the space $SC(X)$ by replacing S^1 everywhere above by X. For the details of the definition of $SC(X)$ we refer the reader to [9].

The subspace $\mathbb{H} = \bigcup_{m=1}^{\infty}\{(x,y) : (x-1/m)^2 + y^2 = 1/m^2\}$ of the Euclidean plane $\mathbb{R}^2$ is called the *Hawaiian earring*. Denote $\theta = (0,0) \in \mathbb{H}$ and let $C(\mathbb{H})$ be the cone over $\mathbb{H}$. We consider $\mathbb{H}$ as the subspace of $C(\mathbb{H})$. A space $\mathcal{G}$ is then defined as the one-point union of two copies of $C(\mathbb{H})$, obtained by identifying two copies of θ at the point θ. This space is a well-known example of a non-contractible space which is a one-point union of contractible spaces – Griffiths was the first to investigate this kind of spaces [11, p.190], where he also acknowleges ideas by James. The fact that $\mathcal{G}$ is aspherical was proved in [8]. For further information of this space and its generalizations we refer the reader to [4, 6, 7].

Throughout the paper, we shall denote the singular homology with integer coefficients by $H_*()$.

3. On nonasphericity of $SC(S^1)$ and $SC(X)$

Obviously, $SC(S^1)$ is a cell-like Peano continuum. It was shown in [9] that this space is simply connected. Therefore it suffices to show that $SC(S^1)$ is nonaspherical. In order to prove this it certainly suffices to verify that there exists a nontrivial 2-dimensional singular cycle in $SC(S^1)$. We shall prove this as a corollary of the following general result – Theorem 3.1 below – in the sense of [9]. Our notation for $SC(X)$ is the same as in [9].

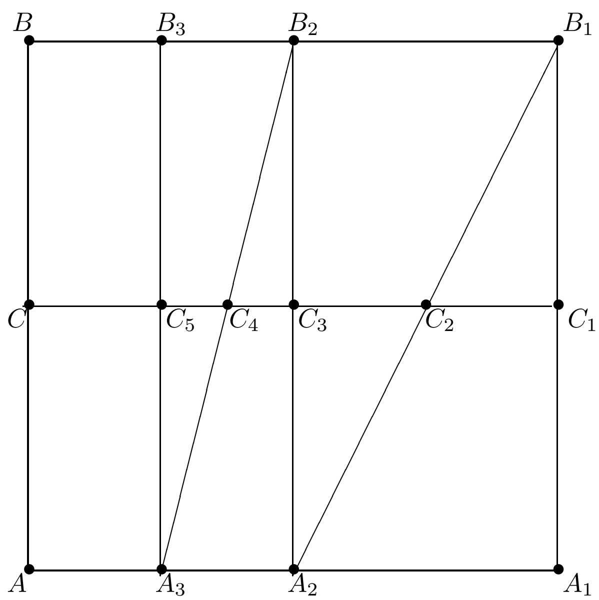

Figure 1

Consider Figure 1: the piecewise linear line $A_1B_1A_2B_2\cdots$ with the segment AB in this figure is the PL Topologist sine curve which was used to build $SC(X)$, i.e. along which we attached the "infinite tube".

Theorem 3.1. *Let X be any path-connected space. Then the following assertions hold:*

(1) *If X is not simply connected, then $H_2(SC(X))$ is not trivial; and*

(2) *If* $\pi_1(X)$ *and* $\pi_2(X)$ *are trivial, then* $H_2(SC(X))$ *is also trivial.*

Corollary 3.2. *The space* $SC(S^1)$ *is a nonaspherical cell-like 2-dimensional simply connected Peano continuum.*

For the proof of Theorem 3.1, we recall a notion of the free σ-product of groups and a lemma from [5]. Let (X_i, x_i) be any family of pointed spaces such that $X_i \cap X_j = \emptyset$, for $i \neq j$. The underlying set of a pointed space $(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ is the union of all X_i s obtained by identifying all x_i to a point x^* and the topology is defined by specifying the neighborhood bases as follows:

(1) If $x \in X_i \setminus \{x_i\}$, then the neighborhood base of x in $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$ is the one of X_i;
(2) The point x^* has a neighborhood base, each element of which is of the form: $\widetilde{\bigvee}_{i\in I\setminus F}(X_i, x_i) \vee \bigvee_{j\in F} U_j$, where F is a finite subset of I and each U_j is an open neighborhood of x_j in X_j for $j \in F$.

Lemma 3.3. [5, Theorem A.1] *Let* X_i *be locally simply-connected and first countable at* x_n *for each* i*. Then*

$$\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \simeq \times^{\sigma}_{i\in I}\pi_1(X_i, x_i).$$

In particular $I = \mathbb{N}$*,*

$$\pi_1(\widetilde{\bigvee}_{n\in\mathbb{N}}(X_n, x_n), x^*) \simeq \times_{n\in\mathbb{N}}\pi_1(X_n, x_n).$$

We also need basic descriptions of paths and loops. A loop $f : \mathbb{I} \to X$ is a continuous map with $f(0) = f(1)$. For a loop f, f^- denotes the loop defined by: $f^-(t) = f(1-t)$. For loops f, g with the same base point, the concatenation fg is a loop defined by: $fg(t) = f(2t)$ for $0 \le t \le 1/2$ and $fg(t) = g(2t-1)$ for $1/2 \le t \le 1$. We denote the homotopy class relative to end points of a loop f by $[f]$ and the homology class of f by $[f]_s$.

Proof of Theorem 3.1. Let p be the natural projection of $SC(X)$ onto $\mathbb{I}^2$ which we consider as a subspace of the plane $\mathbb{R}^2$.

Let $Y_0 = p^{-1}(\mathbb{I} \times [0, 2/3))$ and $Y_1 = p^{-1}(\mathbb{I} \times (1/3, 1])$. Then $SC(X) = Y_0 \cup Y_1$ and $Y_0 \cap Y_1$ is open in $SC(X)$.

Consider the following Mayer-Vietoris homology exact sequence:

$$H_2(SC(X)) \xrightarrow{\partial} H_1(Y_0 \cap Y_1) \xrightarrow{h} H_1(Y_0) \oplus H_1(Y_1).$$

We let $i_0 : Y_0 \cap Y_1 \to Y_0$ and $i_1 : Y_0 \cap Y_1 \to Y_1$ be the inclusion maps. Then $h = i_{0*} - i_{1*}$.

We now present the proof of property (1) above. We first observe that non-injectivity of h implies that $H_2(SC(X))$ is non-trivial.

Since $p^{-1}(\mathbb{I} \times \{0\})$, $p^{-1}(\mathbb{I} \times \{1/2\})$, $p^{-1}(\mathbb{I} \times \{1\})$ are strong deformation retracts of Y_0, $Y_0 \cap Y_1$ and Y_1 respectively, the homotopy types of Y_0, Y_1 and $Y_0 \cap Y_1$ have the same homotopy type as $p^{-1}(\mathbb{I} \times \{0\})$. We denote the deformation retractions by $r_0 : Y_0 \to p^{-1}(\mathbb{I} \times \{0\})$ and $r_1 : Y_1 \to p^{-1}(\mathbb{I} \times \{1\})$.

Choose a point $x^\# \in X$ and form a one point union $(X, x^\#) \vee (\mathbb{I}, 0)$ under the identification of $x^\#$ and 0. Let X_n s be copies of $(X, x^\#) \vee (\mathbb{I}, 0)$ and x_n s copies of $1 \in \mathbb{I}$. Then the space $p^{-1}(\mathbb{I} \times \{0\})$ has the same homotopy type $Y = \widetilde{\bigvee}_{n\in\mathbb{N}}(X_n, x_n)$.

Hence (X_n, x_n) is locally simply connected and first countable at x_n. Lemma 3.3 implies that $\pi_1(Y) \simeq \circledast_{n\in\mathbb{N}}\pi_1(X_n, x_n)$.

Since X is not simply connected, we can find an essential loop f in X whose base point is $x^{\#}$. Observe that $p^{-1}(\{P\})$ is a copy of X for each point P on $A_1B_1A_2B_2\cdots$. A point P on $A_1B_1A_2B_2\cdots$ is written as $P = (x, y)$ for $x, y \in \mathbb{I}$. Define

$$f_P(t) = \begin{cases} (3xt, y), & \text{for } 0 \le t \le 1/3 \\ (P, f(3(t-1/3))), & \text{for } 1/3 \le t \le 2/3 \\ (3(1-t)x, y), & \text{for } 2/3 \le t \le 1. \end{cases}$$

Then for $n \ge 1$, f_{A_n} is a loop in $p^{-1}(\mathbb{I}\times\{0\}) \subseteq Y_0$ with the base point A and f_{B_n} one in $p^{-1}(\mathbb{I}\times\{1\}) \subseteq Y_1$ with the base point B and f_{C_n} one in $p^{-1}(\mathbb{I}\times\{1/2\}) \subseteq Y_0\cap Y_1$ with the base point C respectively. Since the images of f_{C_n} s converge to C, we have two loops $g_0 = f_{C_1}f^-_{C_2}f_{C_3}f^-_{C_4}\cdots$ and $g_1 = f^-_{C_1}f_{C_2}f^-_{C_3}f_{C_4}\cdots$ in $Y_0\cap Y_1$. (These infinite concatenations make sense, since the ranges of loops converge to C.)

Observe that $r_{0*}\circ i_{0*}([f_{C_1}]) = [f_{A_1}]$, $r_{1*}\circ i_{1*}([f_{C_1}]) = [f_{B_1}]$, $r_{0*}\circ i_{0*}([f_{C_{2n}}]) = [f_{A_{n+1}}] = r_{0*}\circ i_{0*}([f_{C_{2n+1}}])$ and $r_{1*}\circ i_{1*}([f_{C_{2n-1}}]) = [B_n] = r_{1*}\circ i_{1*}([f_{C_{2n}}])$ for each natural number n.

Since we have homotopies from $f^-_{A_{n+1}}f_{A_{n+1}}$ to the constant A and the images of the homotopies converge to A, it follows that $r_{0*}\circ i_{0*}([g_1]) = [f_{A_1}]$ and $r_{0*}\circ i_{0*}([g_2]) = [f^-_{A_1}]$. Hence $i_{0*}([g_0g_1]) = e$. Similarly, $r_{1*}\circ i_{1*}([g_0]) = e$ and $r_{1*}\circ i_{1*}([g_1]) = e$ and hence $r_{1*}\circ i_{1*}([g_0g_1]) = e$. Now we have $i_{0*}([g_0g_1]_s) = 0$ and $i_{1*}([g_0g_1]_s) = 0$, i.e. $h([g_0g_1]_s) = 0$.

It suffices to show that $[g_0g_1]_s$ is non-zero, i.e. that $[g_0g_1]$ does not belong to the commutator subgroup of $\pi_1(Y_0\cap Y_1)$. The isomorphism from $\pi_1(Y_0\cap Y_1)$ to $\widetilde{\bigvee}_{n\in\mathbb{N}}(X_n, x_n)$ maps $[g_0g_1]$ to $c_1c_2^{-1}c_3c_4^{-1}\cdots c_1^{-1}c_2c_3^{-1}c_4\cdots$, where c_n is the letter corresponding to $[f_{C_n}]$. To show the conclusion by contradiction, suppose that $c_1c_2^{-1}c_3c_4^{-1}\cdots c_1^{-1}c_2c_3^{-1}c_4\cdots$ belongs to the commutator subgroup. Then, by [5, Lemma 4.11] there exist non-empty reduced words $U_1,\cdots,U_{2m}$ such that $c_1c_2^{-1}c_3c_4^{-1}\cdots c_1^{-1}c_2c_3^{-1}c_4\cdots = U_1\cdots U_{2m}$, where $U_1,\cdots,U_{2m}$ is of the canonical commutator form, i.e. there are j_l, k_l such that $\{j_1,\cdots j_m\}\cup\{k_1,\cdots,k_m\} = \{1,\cdots,2m\}$, $U_{j_l} = U_{k_l}^{-1}$ and the reduced word $c_1c_2^{-1}c_3c_4^{-1}\cdots c_1^{-1}c_2c_3^{-1}c_4\cdots$ is obtained by multiplying the rightmost elements U_i and the leftmost elements of U_{i+1} at most $(2m-1)$-times. Therefore, W_{2m} is of infinite length and is well-ordered from the left to the right, and hence there exists U_i which is of infinite length and is well-ordered from the right to the left. But this is impossible, because $c_1c_2^{-1}c_3c_4^{-1}\cdots c_1^{-1}c_2c_3^{-1}c_4\cdots$ is well-ordered from the left to the right.

Next we show the second statement (2). Suppose that $\pi_1(X)$ and $\pi_2(X)$ are trivial. Consider another part of the Mayer-Vietoris sequence:

$$H_2(Y_0)\oplus H_2(Y_1) \longrightarrow H_2(SC(X)) \xrightarrow{\partial} H_1(Y_0\cap Y_1).$$

By $\pi_1(Y_0\cap Y_1) \simeq \circledast_{n\in\mathbb{N}}\pi_1(X_n, x_n)$, we conclude that $\pi_1(Y_0\cap Y_1)$ is trivial. Hence $H_1(Y_0\cap Y_1)$ is trivial. Since $\pi_1(Y_0)$ is trivial, it follows that $H_2(Y_0)$ is isomorphic to $\pi_2(Y_0)$. Now we have $H_2(Y_0) = \pi_2(Y_0) \simeq \Pi_{n\in\mathbb{N}}\pi_2(X_n, x_n) = \{0\}$ by [7, Theorem 1.1]. Similarly, $H_2(Y_1) = 0$ and $H_2(Y_0)\oplus H_2(Y_1) = \{0\}$. Now the above exact sequence implies that $H_2(SC(X))$ is trivial. □

We denote the commutator $aba^{-1}b^{-1}$ by $[a, b]$.

Alternative proof of Corollary 3.2. For the case $X = S^1$ we take c_n as the generator of the fundamental group of X_{C_n}, which is isomorphic to $\mathbb{Z}$. As in

the preceding proof of Theorem 3.1, it suffices to show that the element $c = c_1c_2^{-1}c_3c_4^{-1}\cdots c_1^{-1}c_2c_3^{-1}c_4\cdots$ does not belong to the commutator subgroup of the group $\pi_1(Y_0 \cap Y_1)$. To prove this by contradiction, suppose that c belongs to the commutator subgroup, i.e. c is a product of m commutators for some m.

Consider natural homomorphism $f : \pi_1(Y_0 \cap Y_1) \to \pi_1(\bigvee_{1\le i\le 2m+2}(X_{C_i}, C_i))$, where $X_{C_i} = S^1$. The group $\pi_1(\bigvee_{1\le i\le 2m+2}(X_{C_i}, C_i))$ is a free group with $2m+2$-generators $\langle c_1, c_2, \cdots, c_{2m+1}, c_{2m+2}\rangle$. We have

$$f(c) = c_1c_2^{-1}\cdots c_{2m+1}c_{2m+2}^{-1}c_1^{-1}c_2\cdots c_{2m+1}^{-1}c_{2m+2}.$$

Let $d_1 = c_1, d_2 = c_2^{-1}$, $d_{2k-1} = c_{2k-2}^{-1}c_{2k-3}\cdots c_2^{-1}c_1c_{2k-1}$ and $d_{2k} = c_{2k}^{-1}c_{2k-1}$.

It is easy to prove by induction the equality $c_1c_2^{-1}\cdots c_{2k-1}c_{2k}^{-1}c_1^{-1}c_2\cdots c_{2k-1}^{-1}c_{2k} = [d_1, d_2]\cdots[d_{2k-1}, d_{2k}]$.

Since $(d_1, d_2, \cdots, d_{2m+1}, d_{2m+2})$ is obtained by a Nielsen transformation [13, p.5] from $(c_1, c_2, \cdots, c_{2m+1}, c_{2m+2})$, the set $\{d_0, d_1, \cdots, d_{2m}, d_{2m+2}\}$ generates the free group $\langle c_1, c_2, \cdots, c_{2m+1}, c_{2m+2}\rangle$. It follows from this and by [13, Proposition 6.8, p.55] (see also [2], p.137) that $f(c)$ cannot be presented as a product of less than $m+1$ commutators. This contradicts our assumption. $\square$

4. A PL Model for $SC(S^1)$ and Some Related Constructions

In this section we demonstrate piecewise linear constructions which are similar to $SC(S^1)$, using parameters for oscillations of a tube. Actually we prove in Theorem 4.3 that they are homotopy equivalent to the point, $SC(S^1)$, or $\mathcal{G}$ depending on their parameters.

For $0 \le y \le 1$ and $\varepsilon \ge 0$ with $0 < y + \varepsilon \le 1$, we construct a space $S(y, \varepsilon) \subseteq \mathbb{R}^3$ as follows. Consider the following points on $\mathbb{I}^2$ for $n \in \mathbb{N}$ (see Figure 2), where we regard $\mathbb{I}^2 \subseteq \mathbb{R}^2$ as $\mathbb{I}^2 \times \{0\}$:

$A_n = (\dfrac{1}{2n-1},\ 0)$, $\quad B_n = (\dfrac{1}{2n},\ 1)$, $\quad C_n = (\dfrac{1}{2n-1},\ y + \dfrac{\varepsilon}{2n-1})$,

$D_n = (\dfrac{1}{2n},\ 1 - y - \dfrac{\varepsilon}{2n})$, $\quad E_n = (\dfrac{1}{2n-1},\ \dfrac{1}{2}(y + \dfrac{\varepsilon}{2n-1}))$,

$F_n = (\dfrac{1}{2n},\ \dfrac{1}{2}(2 - y - \dfrac{\varepsilon}{2n}))$.

Let $\overline{E}_n$ and $\overline{F}_n$ be points on the plane $\{(z, x, y) \in \mathbb{R}^3 |\ z = \frac{1}{2}x\}$ the projections of which to the plane $\mathbb{R}^2$ are points E_n and F_n respectively, i.e.,

$\overline{E}_n = (\dfrac{1}{2n-1},\ \dfrac{1}{2}(y + \dfrac{\varepsilon}{2n-1}),\ \dfrac{1}{2(2n-1)})$, $\quad \overline{F}_n = (\dfrac{1}{2n},\ \dfrac{1}{2}(2 - y - \dfrac{\varepsilon}{2n}),\ \dfrac{1}{4n})$.

Let H_{2n-1} be the convex hull of the points $A_n, B_n, C_n, D_n, \overline{E}_n$ and $\overline{F}_n$ and H_{2n} the convex hull of the points $A_{n+1}, B_n, C_{n+1}, D_n, \overline{F}_n$ and $\overline{E}_{n+1}$.

Let H_∞ be the set $\cup_{n=1}^{\infty} H_n$ and ∂H_∞ its boundary. Let $\Delta A_1C_1\overline{E}_1$ be an open triangle in ∂H_∞. Finally, define $S(y, \varepsilon)$ to be the subspace $(\mathbb{I}^2 \times \{0\}) \cup \partial H_\infty \setminus \Delta A_1C_1\overline{E}_1$ of $\mathbb{R}^3$.

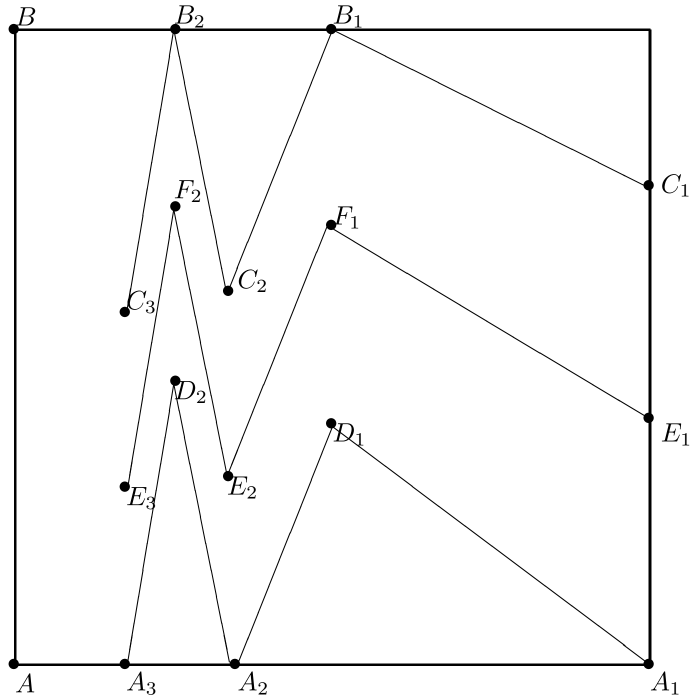

Figure 2: $S(1/2, 1/4)$

The first lemma is easy to prove and we therefore omit its proof.

Lemma 4.1. *Let $\varepsilon, \varepsilon' \in (0,1)$. Then the spaces $S(0,\varepsilon)$ and $S(0,\varepsilon')$ are homeomorphic and $S(0,1)$ is homotopy equivalent to $S(0,\varepsilon)$.*

Lemma 4.2. *If $0 < y \leq 1/2$ and $0 < y + \varepsilon \leq 1$, the space $S(y,\varepsilon)$ is homotopy equivalent to $S(1/2, 0)$.*

Proof. It is easy to see that $S(y,\varepsilon)$ and $S(y,0)$ are homeomorphic and so we only need to prove that $S(y,0)$ for $0 < y < 1/2$ and $S(1/2,0)$ are homotopy equivalent. (Without any loss of generality we may assume that $y = 1/3$.)

Since there might be some confusion regarding the homotopy equivalence, we explain this first. Let A_n, B_n, C_n, D_n,... be the notation for $S(1/2,0)$ and C'_n, D'_n,... be the corresponding notation for $S(1/3,0)$.

If we remove $\{0\} \times \mathbb{I}$ from $S(1/2,0)$ and $S(1/3,0)$, then the resulting spaces are homeomorphic, that is, $S(1/2,0) \setminus \{0\} \times \mathbb{I}$ and $S(1/3,0) \setminus \{0\} \times \mathbb{I}$ are homeomorphic. However, this homeomorphism cannot be extended over to $S(1/2,0)$, since the homeomorphism maps C_n to C'_n and D_n to D'_n, that is, upwards for C_n and downwards for D_n, with respect to the y-coordinate. Conversely, if we construct a homotopy on $S(1/2,0) \setminus \{0\} \times \mathbb{I}$ or $S(1/3,0) \setminus \{0\} \times \mathbb{I}$, whose projection to the y-coordinate only depends on the y-coordinate on the domain, it extends on $SC(1/2,0)$ or $SC(1/3,0)$.

We define $\varphi : S(1/2,0) \to S(1/3,0)$ and $\psi : S(1/2,0) \to S(1/3,0)$ piecewise linearly as follows:

Let $\varphi(x,y,0) = (x,y,0)$ and $\varphi(x,y,z) = (x,y,\varphi_2(x,y,z))$, for $z > 0$, where $\varphi_2(x,y,z) > 0$ if and only if $z > 0$ and there exists $z' > 0$ such that $(x,y,z') \in$

$S(1/3,0)$. Let

$$\psi_1(y) = \begin{cases} 3y/2, & \text{for } 0 \le y \le 1/3 \\ 1/2, & \text{for } 1/3 \le y \le 2/3 \\ 3y/2 - 1/2, & \text{for } 2/3 \le y \le 1. \end{cases}$$

and $\psi(x,y,z) = (\psi_0(x,y,z), \psi_1(y), \psi_2(x,y,z))$, where $\psi_2(x,y,0) = 0$ and $\psi_2(x,y,z) > 0$, for $z > 0$ and $\psi_0(x,y,z)$ is defined as we explain using Figure 3 in the sequel.

Figure 3 demonstrates how $[\frac{1}{2n+1}, \frac{1}{2n}] \times \mathbb{I}$ of $S(1/2,0)$ and $S(1/3,0)$ are mapped by φ and ψ.

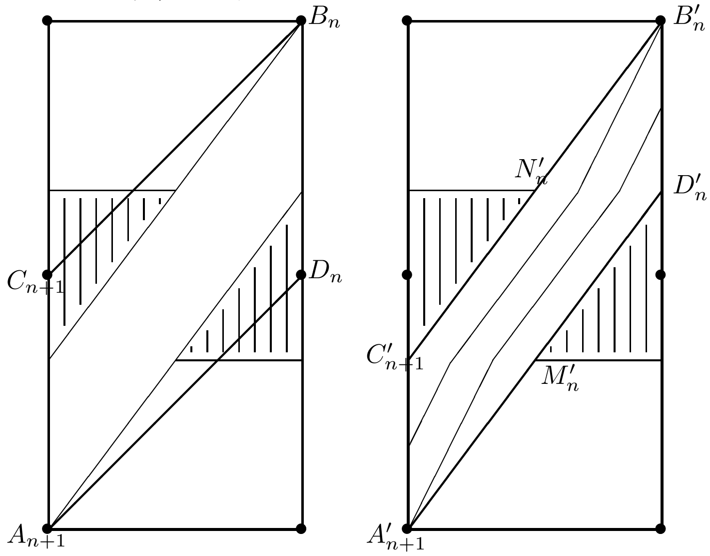

Figure 3 : Parts of $S(1/2,0)$ and $S(1/3,0)$

First we explain the map ψ. The two shadowed triangles are mapped to C_{n+1} or D_n, respectively. Accordingly, the segments $B'_nC'_{n+1}$ and $D'_nA'_{n+1}$ are mapped onto B_nC_{n+1} and D_nA_{n+1} respectively. The segments N'_nD_n and $C'_{n+1}M'_n$ are mapped bijectively to $C_{n+1}D_n$.

Next we explain the map $\varphi\psi$. The two shadowed triangles are mapped to $\varphi(C_{n+1})$ or $\varphi(D_n)$, which are the dotted point. The two bending segments are mapped onto $C'_{n+1}B'_n$ or $A'_{n+1}D'_n$.

Last we explain the map $\psi\varphi$. The two shadowed triangles are mapped to C_{n+1} or D_n. The two segments having slope greater than 1 are mapped to $C_{n+1}B_n$ or $A_{n+1}D_n$.

We have a homotopy $H(x,y,z,t)$ on $S(1/2,0) \setminus (\{0\} \times \mathbb{I})$ such that:

(1) $H(x,y,z,0) = (x,y,z)$ and $H(x,y,z,1) = \psi\varphi(x,y,z)$;
(2) for the y-coordinate $H_1(x,y,z,t)$ of $H(x,y,z,t)$,

$$H_1(x,y,z,t) = \begin{cases} y + yt/2, & \text{for } 0 \le y \le 1/3 \\ y + t/2 - yt, & \text{for } 1/3 \le y \le 2/3 \\ y - t/2 + yt/2, & \text{for } 2/3 \le y \le 1; \end{cases}$$

(3) $H(*,*,*,t)$ maps $p^{-1}([\frac{1}{n+1}, \frac{1}{n}] \times \mathbb{I})$ onto itself for each n.

Then we can extend $H(*,*,*,t)$ to $S(1/2,0)$ uniquely and continuously.

Concerning $S(1/3,0)$ with $\varphi\psi$, we have a homotopy with the same properties as above and we now see that $S(1/2,0)$ and $S(1/3,0)$ are homotopy equivalent. □

Theorem 4.3. *Suppose that $0 \leq y \leq 1$, $\varepsilon \geq 0$ and $0 < y+\varepsilon \leq 1$. Then the following assertions hold:*

1. *For every $1/2 < y \leq 1$, the spaces $S(1,0)$ and $S(y,\varepsilon)$ are contractible;*
2. *For $y=0$, the space $S(y,\varepsilon)$ is homotopy equivalent to $SC(S^1)$; and*
3. *For every $0 < y \leq 1/2$, the space $S(y,\varepsilon)$ is homotopy equivalent to the space $\mathcal{G}$.*

Proof. The statements (1) and (2) are easy to verify. Therefore we shall only prove (3).

By Lemma 4.2, it suffices to show that $S(1/2,1/4)$ is homotopy equivalent to the space $\mathcal{G}$. Let Δ be the half-open triangle, defined as $\Delta = \{(x,y) \mid x \in (0,1], y \in (-x/4+1/2, x/4+1/2)\}$. Then $p^{-1}(\mathbb{I}^2 \setminus \Delta)$ is a strong deformation retract of $S(1/2,1/4)$.

Identifying $\{(x,y) \mid y = a+(1-a)x/4, x \in \mathbb{I}\}$ as one point for $a \in [1/2,1]$ and $\{(x,y) \mid y = a - ax/2, x \in \mathbb{I}\}$ as one point for $a \in [0,1/2]$, we get the quotient space of $p^{-1}(\mathbb{I}^2 \setminus \Delta)$, which is homeomorphic to $\mathcal{G}$. Now the homotopy equivalence between $p^{-1}(\mathbb{I}^2 \setminus \Delta)$ and $\mathcal{G}$ is evident and so $S(1/2,1/4)$ is indeed homotopy equivalent to $\mathcal{G}$. □

Remark 4.4. The space $SC(S^1)$ is simply connected (see [9]), whereas the space $\mathcal{G}$ is not simply connected (see [11]). We remark that $H_2(\mathcal{G}) = \{0\}$, which contrasts with Theorem 3.2.

To show this, we introduce some notation. Since the cone $C(X)$ over the space X is the quotient space of $X \times \mathbb{I}$, obtained by identifying $X \times \{1\}$ to a point, we let $p : X \times \mathbb{I} \to C(X)$ be the canonical projection.

For a subset A of $\mathbb{I}$, let $C_A(X) = p(X \times A) \subset C(X)$. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be copies of the Hawaiian earring $\mathbb{H}$ and $\mathcal{G} = C(\mathbb{H}_1) \vee C(\mathbb{H}_2)$ be the one point union of $C(\mathbb{H}_1)$ and $C(\mathbb{H}_2)$ defined in Section 2. Let X_1 be the disjoint union of $C_{(1/3,1]}(\mathbb{H}_1)$ and $C_{(1/3,1]}(\mathbb{H}_2)$ and X_2 be $C_{[0,2/3)}(\mathbb{H}_1) \vee C_{[0,2/3)}(\mathbb{H}_2)$.

Then $\mathcal{G} = X_1 \cup X_2$ and we have the following part of the Mayer-Vietoris sequence:

$$H_2(X_1) \oplus H_2(X_2) \longrightarrow H_2(\mathcal{G}) \xrightarrow{\partial} H_1(X_1 \cap X_2) \xrightarrow{h} H_1(X_1) \oplus H_1(X_2).$$

Obviously, $H_2(X_1) = \{0\}$. Since X_2 is homotopy equivalent to $\mathbb{H}_1 \vee \mathbb{H}_2$ which is a 1-dimensional compact metric space, $H_2(X_2)$ is trivial [3]. Hence ∂ is injective. We observe that $X_1 \cap X_2$ is the disjoint union of $C_{(1/3,2/3)}(\mathbb{H}_1)$ and $C_{(1/3,2/3)}(\mathbb{H}_2)$.

Since $C_{[1/3,2/3)}(\mathbb{H}_1)$ and $C_{[1/3,2/3)}(\mathbb{H}_2)$ are retracts of $C_{[0,2/3)}(\mathbb{H}_1) \vee C_{[0,2/3)}(\mathbb{H}_2)$ and are homotopy equivalent to $C_{(1/3,2/3)}(\mathbb{H}_1)$ and $C_{(1/3,2/3)}(\mathbb{H}_2)$ respectively, it follows that h is injective. Therefore we obtain that $H_2(\mathcal{G}) = \{0\}$.

Problem 4.5. *Does there exist a finite-dimensional non-contractible Peano continuum all homotopy groups of which are trivial?*

Remark 4.6. *Recently we have strengthened Theorem 3.1(2) by proving the following: If X is simply connected, then $\pi_2(SC(X))$ is trivial. We have proved earlier that $SC(X)$ is also simply connected* [9]*. Therefore by Theorem 3.1(1) the following statements are equivalent for any path-connected space X:*

1. *X is simply connected;*

(2) $\pi_2(SC(X))$ *is trivial; and*
(3) $H_2(SC(X))$ *is trivial.*

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